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PROPERTIES OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT

Two integral operators P^{α} and Q_{β}^{α} for analytic functions in the open unit disk are introduced. The object of the present paper is to derive some properties of integral operators P^{α} and Q_{β}^{α}.

I. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1\}$. Recently, Jung, Kim and Srivastava [4] have introduced the following one-parameter families of integral operators:

$$P^{\alpha}f = P^{\alpha}f(z) = \frac{2^{\alpha}}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\,dt \qquad (\alpha > 0), \tag{1.2}$$

$$Q_{\beta}^{\alpha}f = Q_{\beta}^{\alpha}f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt \qquad (\alpha > 0,\ \beta > -1), \tag{1.3}$$

and

$$J_{\alpha}f = J_{\alpha}f(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z t^{\alpha-1} f(t)\,dt \qquad (\alpha > -1), \tag{1.4}$$

where $\Gamma(\alpha)$ is the familiar Gamma function, and (in general)

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)} = \binom{\alpha}{\alpha-\beta}. \tag{1.5}$$

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For $\alpha \in \mathbb{N} = \{1,2,3,\ldots\}$, the operators P^α, Q_1^α, and J_α were considered by Bernardi ([1], [2]). Further, for a real number $\alpha > -1$, the operator J_α was used by Owa and Srivastava [6], and by Srivastava and Owa ([7], [8]).

REMARK. For $f(z) \in \mathcal{A}$ given by (1.1), Jung, Kim and Srivastava [4] have shown that

$$P^\alpha f(z) = z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\alpha a_n z^n \qquad (\alpha > 0), \tag{1.6}$$

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \qquad (\alpha > 0,\ \beta > -1), \tag{1.7}$$

and

$$J_\alpha f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n \qquad (\alpha > -1). \tag{1.8}$$

By virtue of (1.6) and (1.8), we see that

$$J_\alpha f(z) = Q_\alpha^1 f(z) \qquad (\alpha > -1). \tag{1.9}$$

2. AN APPLICATION OF MILLER-MOCANU LEMMA

To derive some properties of operators, we have to recall here the following lemma due to Miller and Mocanu [5].

LEMMA I. Let $w(u,v)$ be a complex valued function,

$w: \mathbb{D} \longrightarrow \mathbb{C}$, $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane),

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $w(u,v)$ satisfies the following conditions:

(i) $w(u,v)$ is continuous in $\mathbb{D}$;

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\{w(1,0)\} > 0$;

(iii) $\mathrm{Re}\{w(iu_2,v_1)\} \leqq 0$ for all $(iu_2,v_1) \in \mathbb{D}$ and such that $v_1 \leqq -(1+u_2^2)/2$.

Let $p(z)$ be regular in $\mathcal{U}$ and $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ such that $(p(z), zp'(z)) \in \mathcal{D}$ for all $z \in \mathcal{U}$. If $\mathrm{Re}\{w(p(z), zp'(z))\} > 0$ $(z \in \mathcal{U})$, then $\mathrm{Re}\{p(z)\} > 0$ $(z \in \mathcal{U})$.

Applying the above lemma, we derive

THEOREM 1. If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} > \beta \qquad (\alpha > 2;\ z \in \mathcal{U}) \tag{2.1}$$

for some β $(\beta < 1)$, then

$$\mathrm{Re}\left\{ \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \right\} > \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \qquad (z \in \mathcal{U}). \tag{2.2}$$

PROOF. Noting that

$$z(P^{\alpha}f(z))' = 2P^{\alpha-1}f(z) - P^{\alpha}f(z) \qquad (\alpha > 1), \tag{2.3}$$

we have

$$\frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} = 2\,\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} - 1 \qquad (\alpha > 1). \tag{2.4}$$

Define the function $p(z)$ by

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} = \gamma + (1 - \gamma)p(z) \tag{2.5}$$

with

$$\gamma = \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8}. \tag{2.6}$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in the open unit disk $\mathcal{U}$. Since

$$\frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)} - \frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} = \frac{(1 - \gamma)zp'(z)}{\gamma + (1 - \gamma)p(z)}, \tag{2.7}$$

or

$$(2.8) \qquad \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} + \frac{(1-\gamma)zp'(z)}{2\{\gamma + (1-\gamma)p(z)\}},$$

we have

$$(2.9) \qquad \mathrm{Re}\left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} - \beta \right\}$$

$$= \mathrm{Re}\left\{ \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{2\{\gamma + (1-\gamma)p(z)\}} - \beta \right\}$$

$$> 0.$$

Therefore, if we define the function $w(u,v)$ by

$$(2.10) \qquad w(u,v) = \gamma - \beta + (1-\gamma)u + \frac{(1-\gamma)v}{2\{\gamma + (1-\gamma)u\}},$$

then we see that

(i) $w(u,v)$ is continuous in $\mathbb{D} = \left[\mathbb{C} - \{\frac{\gamma}{\gamma-1}\}\right] \times \mathbb{C}$;

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\{w(1,0)\} = 1 - \beta > 0$;

(iii) for all $(iu_2, v_1) \in \mathbb{D}$ and such that $v_1 \leqq -(1 + u_2^2)/2$,

$$\mathrm{Re}\{w(iu_2, v_1)\} = \gamma - \beta + \frac{\gamma(1-\gamma)v_1}{2\{\gamma^2 + (1-\gamma)^2 u_2^2\}}$$

$$\leqq \gamma - \beta - \frac{\gamma(1-\gamma)(1+u_2^2)}{4\{\gamma^2 + (1-\gamma)^2 u_2^2\}}$$

$$= - \frac{(1-\gamma)(\gamma + 4(1-\gamma)\beta)}{4\{\gamma^2 + (1-\gamma)^2 u_2^2\}} u_2^2$$

$$\leqq 0.$$

This implies that the function $w(u,v)$ satisfies the conditions in Lemma 1. Thus, applying Lemma 1, we conclude that

$$(2.11) \qquad \mathrm{Re}\left\{ \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \right\} > \gamma = \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \qquad (z \in \mathbb{U}).$$

Taking the special values for β in Theorem 1, we have

COROLLARY I. Let $f(z)$ be in the class $\mathcal{A}$. Then

(i) $$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{2} \quad (z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{4} \quad (z \in \mathcal{U}),$$

(ii) $$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{4} \quad (z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{\sqrt{5}-1}{4} \quad (z \in \mathcal{U}),$$

(iii) $$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > 0 \quad (z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{\sqrt{17}-1}{8} \quad (z \in \mathcal{U}),$$

(iv) $$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{4} \quad (z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{2} \quad (z \in \mathcal{U}),$$

and

(v) $$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{2} \quad (z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{\sqrt{17}+1}{8} \quad (z \in \mathcal{U}).$$

Next, we prove

THEOREM 2. If $f(z) \in \mathcal{A}$ satisfies

$$(2.12)\qquad \operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}\right\} > \gamma \qquad (\alpha > 2,\ \beta > -1; z \in \mathrm{U})$$

for some γ $((\alpha+\beta-3)/2(\alpha+\beta-1) \leq \gamma < 1)$, then

$$(2.13)\qquad \operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\} > \delta \qquad (z \in \mathrm{U}),$$

where

$$(2.14)\qquad \delta = \frac{1 + 2\gamma(\alpha+\beta-1) + \sqrt{(1+2\gamma(\alpha+\beta-1))^2 + 8(\alpha+\beta)}}{4(\alpha + \beta)}.$$

PROOF. By the definition of $Q_\beta^\alpha f(z)$, we know that

$$(2.15)\qquad z(Q_\beta^\alpha f(z))' = (\alpha + \beta)Q_\beta^{\alpha-1}f(z) - (\alpha + \beta - 1)Q_\beta^\alpha f(z) \qquad (\alpha > 1,\ \beta > -1),$$

so that

$$(2.16)\qquad \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} = (\alpha + \beta)\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^\alpha f(z)} - (\alpha + \beta - 1).$$

We define the function $p(z)$ by

$$(2.17)\qquad \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^\alpha f(z)} = \delta + (1 - \delta)p(z).$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in U. Making use of the logarithmic differentiations in both sides of (2.17), we have

$$(2.18)\qquad \frac{z(Q_\beta^{\alpha-1}f(z))'}{Q_\beta^{\alpha-1}f(z)} = \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}.$$

Applying (2.15) to (2.18), we obtain that

$$(2.19)\qquad \frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}$$

$$= \frac{1}{\alpha + \beta - 1}\left\{(\alpha + \beta)\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^\alpha f(z)} - 1 + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}\right\}$$

$$= \frac{1}{\alpha + \beta - 1}\left\{\delta(\alpha + \beta) - 1 + (\alpha + \beta)(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}\right\},$$

that is, that

$$(2.20) \quad \mathrm{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} - \gamma\right\}$$

$$= \frac{1}{\alpha + \beta - 1}\mathrm{Re}\left\{\delta(\alpha + \beta) - 1 + (\alpha + \beta)(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}\right\} - \gamma > 0.$$

Now, we let

$$(2.21) \quad w(u,v) = \frac{1}{\alpha+\beta-1}\left\{\delta(\alpha+\beta) - 1 + (\alpha+\beta)(1-\delta)u + \frac{(1-\delta)v}{\delta+(1-\delta)u}\right\} - \gamma$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then $w(u,v)$ satisfies that

(i) $w(u,v)$ is continuous in $\mathbb{D} = \left(\mathbb{C}-\{\frac{\delta}{\delta-1}\}\right) \times \mathbb{C}$;

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\{w(1,0)\} = 1 - \gamma > 0$;

(iii) for all $(iu_2,v_1) \in \mathbb{D}$ and such that $v_1 \leqq -(1 + u_2^2)/2$,

$$\mathrm{Re}\{w(iu_2,v_1)\} = \frac{1}{\alpha+\beta-1}\left\{\delta(\alpha+\beta) - 1 +\right.$$

$$= \frac{1}{\alpha+\beta-1}\left\{\delta(\alpha+\beta) - 1 + \frac{\delta(1-\delta)v_1}{\delta^2 + (1-\delta)^2u_2^2}\right\} - \gamma$$

$$\leqq \frac{1}{\alpha+\beta-1}\left\{\delta(\alpha+\beta) - 1 - \gamma(\alpha+\beta-1) - \frac{\delta(1-\delta)(1+u_2^2)}{2\{\delta^2 + (1-\delta)^2u_2^2\}}\right\}$$

$$\leqq 0.$$

Thus, the function $w(u,v)$ satisfies the conditions in Lemma 1. This shows that $\mathrm{Re}\{p(z)\} > 0$ $(z \in \mathbb{U})$, or

$$\text{(2.22)} \qquad \mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} \right\} > \delta \qquad (z \in U).$$

If we take $\gamma = (\alpha+\beta-3)/2(\alpha+\beta-1)$ in Theorem 2, then we have

COROLLARY 2. If $f(z) \in A$ satisfies

$$\mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} \right\} > \frac{\alpha + \beta - 3}{2(\alpha + \beta - 1)} \qquad (\alpha > 2,\ \beta > -1; z \in U),$$

then

$$\mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} \right\} > \frac{1}{2} \qquad (z \in U).$$

Further, letting $\alpha = 2 - \beta$ and $\gamma = 1/2$ in Theorem 2, we have

COROLLARY 3. If $f(z) \in A$ satisfies

$$\mathrm{Re}\left\{ \frac{Q_\beta^{-\beta}f(z)}{Q_\beta^{1-\beta}f(z)} \right\} > \frac{1}{2} \qquad (\beta > -1;\ z \in U),$$

then

$$\mathrm{Re}\left\{ \frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)} \right\} > \frac{1 + \sqrt{5}}{4} \qquad (z \in U).$$

3. AN APPLICATION OF JACK'S LEMMA

We need the following lemma due to Jack [3] (also, due to Miller and Mocanu [5]).

LEMMA 2. Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we can write

$$\text{(3.1)} \qquad z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geqq 1$.

Applying Lemma 2 for the operator P^α, we have

THEOREM 3. If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \beta \qquad (\alpha > 2,\ \beta \leq 1/4; z \in \mathcal{U}), \tag{3.2}$$

then

$$\mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \gamma \qquad (z \in \mathcal{U}), \tag{3.3}$$

where

$$\gamma = \frac{3 + 4\beta + \sqrt{16\beta^2 - 40\beta + 9}}{8}. \tag{3.4}$$

PROOF. Defining the function $w(z)$ by

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} = \frac{1 - (1 - 2\gamma)w(z)}{1 + w(z)}, \tag{3.5}$$

we see that $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$. It follows from (3.5) that

$$\frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)} = \frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} - \frac{(1 - 2\gamma)zw'(z)}{1 - (1 - 2\gamma)w(z)} - \frac{zw'(z)}{1 + w(z)}. \tag{3.6}$$

Using (2.3), we obtain that

$$\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{1 - (1 - 2\gamma)w(z)}{1 + w(z)} - \frac{1}{2}\frac{zw'(z)}{w(z)}\left(\frac{(1 - 2\gamma)w(z)}{1 - (1 - 2\gamma)w(z)} + \frac{w(z)}{1 + w(z)}\right). \tag{3.7}$$

If we suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq -1),$$

then Lemma 2 gives us that

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ $(\theta \neq \pi)$, we have

$$(3.8) \quad \mathrm{Re}\left\{ \frac{P^{\alpha-2}f(z_0)}{P^{\alpha-1}f(z_0)} \right\}$$

$$= \mathrm{Re}\left\{ \frac{1 - (1 - 2\gamma)w(z_0)}{1 + w(z_0)} - \frac{k}{2}\left[\frac{(1 - 2\gamma)w(z_0)}{1 - (1 - 2\gamma)w(z_0)} + \frac{w(z_0)}{1 + w(z_0)} \right]\right\}$$

$$= \mathrm{Re}\left\{ \frac{1 - (1 - 2\gamma)e^{i\theta}}{1 + e^{i\theta}} - \frac{k}{2}\left[\frac{(1 - 2\gamma)e^{i\theta}}{1 - (1 - 2\gamma)e^{i\theta}} + \frac{e^{i\theta}}{1 + e^{i\theta}} \right]\right\}$$

$$= \gamma - \frac{k}{2}\left[\frac{(1 - 2\gamma)(\cos\theta - (1 - 2\gamma))}{1 + (1 - 2\gamma)^2 - 2(1 - 2\gamma)\cos\theta} + \frac{1}{2} \right]$$

$$\leqq \gamma - \frac{k}{2}\left[\frac{1}{2} - \frac{1 - 2\gamma}{2(1 - \gamma)} \right]$$

$$\leqq \frac{\gamma(3 - 4\gamma)}{4(1 - \gamma)} = \beta.$$

This contradicts our condition (3.2). Therefore, we have

$$(3.9) \qquad |w(z)| = \left| \frac{\dfrac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} - 1}{\dfrac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} + (1 - 2\gamma)} \right| < 1 \qquad (z \in \mathcal{U}),$$

which implies (3.3). This completes the proof of Theorem 3.

Taking $\beta = 0$ and $\beta = 1/4$ in Theorem 3, we have

COROLLARY 3. Let $f(z)$ be in the class $\mathcal{A}$. Then

$$\mathrm{Re}\left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} > 0 \quad (\alpha > 2; z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{ \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \right\} > \frac{3}{4} \qquad (z \in \mathcal{U}),$$

and

$$\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{4} \qquad (\alpha > 2; z \in \mathcal{U})$$

$$\Longrightarrow \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}).$$

Finally, we prove

THEOREM 4. If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}\right\} > \gamma \qquad (\alpha > 2,\ \beta > -1;\ z \in \mathcal{U}) \tag{3.10}$$

for some γ ($\gamma < 1$), then

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\} > \delta \qquad (z \in \mathcal{U}), \tag{3.11}$$

where δ ($0 \leqq \delta < 1$) is the smallest positive root of the equation

$$2(\alpha+\beta)\delta^2 - \{2(\alpha+\beta)(\gamma+1) - (2\gamma-1)\}\delta + 2\{(\alpha+\beta-1)\gamma + 1\} = 0. \tag{3.12}$$

PROOF. Define the function $w(z)$ by

$$\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} = \frac{1 - (1 - 2\delta)w(z)}{1 + w(z)}, \tag{3.13}$$

we obtain that

$$\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} = \frac{1}{\alpha+\beta-1}\left\{(\alpha+\beta)\,\frac{1 - (1-2\delta)w(z)}{1 + w(z)} - 1 - \frac{zw'(z)}{w(z)}\left(\frac{(1-2\delta)w(z)}{1 - (1-2\delta)w(z)} + \frac{w(z)}{1 + w(z)}\right)\right\}. \tag{3.14}$$

Therefore, supposing that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq -1),$$

we see that

$$\text{(3.15)} \quad \mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-2}f(z_0)}{Q_\beta^{\alpha-1}f(z_0)} \right\} \leqq \frac{1}{\alpha+\beta-1}\left\{(\alpha+\beta)\delta - 1 - \frac{\delta}{2(1-\delta)}\right\}$$

$$= \gamma.$$

This completes the assertion of Theorem 4.

Making $\gamma = 0$ in Theorem 4, we have

COROLLARY 4. If $f(z) \in \mathcal{A}$ satisfies

$$\text{(3.16)} \qquad \mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} \right\} > 0 \qquad (\alpha > 2,\ \beta > -1; z \in \mathcal{U}),$$

then

$$\text{(3.17)} \qquad \mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} \right\} > \frac{-1}{\alpha+\beta-1} \qquad (z \in \mathcal{U}).$$

Letting $\gamma = 1/2$ in Theorem 4, we have

COROLLARY 5. If $f(z) \in \mathcal{A}$ satisfies

$$\text{(3.18)} \qquad \mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} \right\} > \frac{1}{2} \qquad (\alpha > 2,\ \beta > -1; z \in \mathcal{U}),$$

then

$$\text{(3.19)} \qquad \mathrm{Re}\left\{ \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} \right\} > \frac{\alpha+\beta-3}{\alpha+\beta-1} \qquad (z \in \mathcal{U}).$$

Further, if $\alpha = 3 - \beta$, we have

$$\mathrm{Re}\left\{ \frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)} \right\} > \frac{1}{2} \qquad (\beta > -1;\ z \in \mathcal{U})$$

$$\Longrightarrow \mathrm{Re}\left\{ \frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)} \right\} > 0 \qquad (z \in \mathcal{U}).$$

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REFERENCES

[1] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135(1969), 429 - 446.

[2] S. D. Bernardi, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 24(1970), 312 - 318.

[3] I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc. 3(1971), 469 - 474.

[4] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, to appear.

[5] S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(1978), 289 - 305.

[6] S. Owa and H. M. Srivastava, Some applications of the generalized Libera operator, Proc. Japan Acad. 62(1986), 125 - 128.

[7] H. M. Srivastava and S. Owa, New characterization of certain starlike and convex generalized hypergeometric functions, J. Nat. Acad. Math. India 3(1985), 198 - 202.

[8] H. M. Srivastava and S. Owa, A certain one-parameter additive family of operators defined on analytic functions, J. Math. Anal. Appl. 118(1986), 80 - 87.

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